Construction of the 2-dimensional Grosse-Wulkenhaar Model

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Abstract

In this paper we construct the noncommutative Grosse-Wulkenhaar model on 2-dimensional Moyal plane with the method of loop vertex expansion. We treat renormalization with this new tool, adapt Nelson's argument and prove Borel summability of the perturbation series. This is the first non-commutative quantum field theory model to be built in a non-perturbative sense.

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1 Introduction

Quantum field theories on noncommutative space time became popular after the discovery that they may arise as effective regimes of string theory either due to the compactification [1] or due to the presence of the Green-Schwarz B field for open strings [2, 3]. The simplest non-commutative space is the Moyal space, which could be considered also as a low energy limit of open string theory. However the quantum field theories on the Moyal space are non-renormalizable due to a phenomenon called UV/IR mixing, namely when we integrate out the high scale fields for non planar graphs, there exist still infrared divergences and the divergent terms cannot be compensated by counterterms [4].

Several years ago H. Grosse and R. Wulkenhaar made a breakthrough by introducing a harmonic oscillator term in the propagator so that the theory fully obeys a new symmetry

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called the Langmann-Szabo duality. They have proved in a series of papers [5, 6, 7] (see also [8]) that the noncommutative $\phi_4^{\star 4}$ field theory possessing the Langmann-Szabo duality (which we call the GW_4 model hereafter) is perturbatively renormalizable to all orders.

After their work many other QFT models on Moyal space [9, 10, 11, 14, 12, 13, 15] or degenerate Moyal space [16, 17] have also been shown to be *perturbatively* renormalizable. More details could be found in [20, 21, 22].

The GW_4 model is not only perturbatively renormalizable but also asymptotically safe due to the vanishing of the β function in the ultraviolet regime [24], [25], [26], and the renormalization flow of the coupling constant is bounded, which means that it is even better behaved than its commutative counterpart. In the commutative case we either have the Landau ghost for the ϕ_4^4 theory and QED in the ultraviolet regime or we have confinement for non-Abelian gauge theory in the infrared regime. This makes the non-perturbative construction of these commutative field theories either difficult or impossible; it also means that GW_4 is a prime natural candidate for a full construction of a four dimensional just renormalizable quantum field theory [28].

Constructive field theory or constructive renormalization theory builds the exact Green's functions whose Taylor expansions correspond to perturbative quantum field theory. The traditional techniques for Bosonic constructive theories are the cluster and Mayer expansions which divide the space into cubes and require locality of the interaction. However due to the noncommutativity of the coordinates in Moyal plane, the naive division of space into cubes seems unsuited in this case. What's more, due to the non-locality of the interaction vertex, it is not clear to which cube an interaction vertex should belong. So it seems very difficult, if not impossible, to construct the Grosse-Wulkenhaar model with the traditional methods of cluster and Mayer expansion.

In this paper we shall construct the Grosse-Wulkenhaar model in the 2 dimensional Moyal plane or GW_2, as a warm-up towards building non-perturbatively the full GW_4 model. The method we use is the Loop Vertex Expansion (LVE) which was invented precisely for overcoming these difficulties of the traditional methods [27] and has been applied also to other QFT models with cutoffs [29, 30, 31]. This method has been tested first on the ordinary ϕ_2^4 commutative case in a companion paper [32]. Another approach towards the construction of GW_4, based on the combination of a Ward identity and Schwinger-Dyson equations, is given in [33]. A non-perturbative numerical study of the noncommutative quantum field theory model is given in [18].

The loop vertex expansion is a combination of the intermediate field techniques with the BKAR forest formula [39, 40]. In the case of a $\lambda\phi^4$ interaction, the first step is to introduce an intermediate field σ and integrate out the original field ϕ. After that we get a non local interaction $(1/2)\mathrm{Tr}\log(1+i\sqrt{\lambda}C^{1/2}\sigma C^{1/2})$, where Tr means summing over all indices for the matrix model (or integrating over the whole space for the commutative case) while keeping the cyclic order, and C is the covariance of the original fields ϕ. After this step we have a new representation of the same theory where the dynamical variables are the intermediate fields σ. The new features are that the interaction vertex becomes non-local (even when the initial vertex, like in ordinary ϕ^4 was local) while the covariance of the σ fields becomes ultralocal, which means that the σ fields have trivial propagation.

Then we apply the BKAR tree formula to obtain the (unrenormalized) connected Green's functions. But this is not the whole story. Some parts of the Wick ordering counterterms

can be easily compensated with the corresponding terms within the LVE formalism. But the Green's functions obtained after the LVE are still divergent due to other remaining counter-terms from the Wick ordering of the interaction which correspond to terms which are not explicitly visible within the LVE. This is why we need a second expansion to generate terms called *inner tadpoles* and to compensate them with their associated counter-terms. We call this second expansion the cleaning expansion; it was introduced and applied to the commutative ϕ_2^4 case in [32].

After this cleaning expansion, using analogs of the Nelson's bounds for ϕ_2^4 (see [28] and references therein), we obtain convergence of the expansion hence rigorous construction of the Green's functions and we can prove their Borel summability. In fact due to the harmonic Grosse-Wulkenhaar potential these bounds are *easier* in the GW_2 case.

This paper is organized as follows. In section 2 we introduce the basic properties of the noncommutative Moyal space, the matrix basis for functions defined on this space, the GW_2 model in the matrix basis and the combinatorial factors for its Wick ordering. In section 3 we introduce the intermediate field techniques and the BKAR tree formula. In section 4 we consider the amplitude, the graph representation and the dual representation of the connected functions obtained by the LVE. In section 5 we define the cleaning expansion. In section 6 we consider the renormalization. Sections 7 and 8 are devoted to the resummation of the counter-terms and the Borel-sumability. Two examples for the first and second order perturbative renormalizations are given in the Appendix.

2 Moyal space and Grosse-Wulkenhaar Model

2.1 The Moyal space

The D-dimensional Moyal space $\mathbb{R}^D_\theta$ for D even is generated by the non-commutative coordinates x^μ that obey the commutation relation $[x^\mu, x^\nu] = i\Theta^{\mu\nu}$, where Θ is a $D\times D$ non-degenerate skew-symmetric matrix such that $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$. It is the simplest and best studied model of non-commutative space (see [19, 20] for more details).

Consider the Moyal algebra of smooth and rapidly decreasing functions $\mathcal{S}(\mathbb{R}^D)$ defined on $\mathbb{R}^D_\theta$. $\mathcal{S}(\mathbb{R}^D)$ is equipped with the non-commutative *Groenwald-Moyal* product defined by: $\forall f, g \in \mathcal{S}_D := \mathcal{S}(\mathbb{R}^D)$,

$$
\begin{aligned}
(f \star_\Theta g)(x) &= \int_{\mathbb{R}^D} \frac{d^D k}{(2\pi)^D} d^D y\, f(x + \tfrac{1}{2}\Theta\cdot k) g(x+y) e^{\imath k\cdot y} && (1)\\
&= \frac{1}{\pi^D |\det \Theta|} \int_{\mathbb{R}^D} d^D y d^D z\, f(x+y) g(x+z) e^{-2\imath y\Theta^{-1} z} . && (2)
\end{aligned}
$$

We now set $D = 2$ in this paper. We could define the creation and annihilation operators in terms of the coordinates of the Moyal plane $\mathbb{R}^2_\theta$ [6, 7, 19]:

$$
\begin{aligned}
a &= \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2) , & \bar{a} &= \frac{1}{\sqrt{2}}(x_1 - \mathrm{i}x_2) ,\\
\frac{\partial}{\partial a} &= \frac{1}{\sqrt{2}}(\partial_1 - \mathrm{i}\partial_2) , & \frac{\partial}{\partial \bar{a}} &= \frac{1}{\sqrt{2}}(\partial_1 + \mathrm{i}\partial_2) . && (3)
\end{aligned}
$$

such that for any function $f \in \mathcal{S}_D$ we have

$$(a \star f)(x) = a(x)f(x) + \frac{\theta_1}{2}\frac{\partial f}{\partial \bar{a}}(x) \; , \qquad (f \star a)(x) = a(x)f(x) - \frac{\theta_1}{2}\frac{\partial f}{\partial \bar{a}}(x) \; ,$$
$$(\bar{a} \star f)(x) = \bar{a}(x)f(x) - \frac{\theta_1}{2}\frac{\partial f}{\partial a}(x) \; , \qquad (f \star \bar{a})(x) = \bar{a}(x)f(x) + \frac{\theta_1}{2}\frac{\partial f}{\partial a}(x) \; . \tag{4}$$

With the creaton and annihilation operators we could build the matrix basis for the functions $\phi(x) \in \mathcal{S}(\mathbb{R}^2)$ defined on $\mathbb{R}^D_\theta$, such that:

$$\phi(x) = \sum_{mn} \phi_{mn} f_{mn}(x), \tag{5}$$

where the matrix basis $f_{mn}(x)$ is defined as:

$$\begin{aligned} f_{mn}(x) &:= \frac{1}{\sqrt{n!m!\,\theta_1^{m+n}}}\,\bar{a}^{\star m} \star f_0 \star a^{\star n} \qquad (6)\\ &= \frac{1}{\sqrt{n!m!\,\theta_1^{m+n}}} \sum_{k=0}^{\min(m,n)} (-1)^k \binom{m}{k}\binom{n}{k}\, k!\, 2^{m+n-2k}\, \theta^k\, \bar{a}^{m-k}\, a^{n-k} f_0 \; , \end{aligned}$$

and $f_0(x) = 2\mathrm{e}^{-\frac{1}{\theta}(x_1^2+x_2^2)}$.

The matrix basis defined above has the following properties:

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x) \; . \tag{7}$$

and

$$\int d^2x f_{mn}(x) = 2\pi\theta\delta_{mn}. \tag{8}$$

2.2 Ribbon graphs

Matrix models or quasi-matrix models use *ribbon* graphs, which were first introduced by 't Hooft for QCD [36]. A ribbon graph $G = (V, E)$ is an orientable surface with boundary represented as a set of closed disks, also called vertices, V and a set of ribbons, also called edges E, such that all disks are attached to each other by the ribbons glued to their boundaries. Figure 1 is an example of a non-planar ribbon graph. The interested reader could go to [37, 38] for a more general introduction.

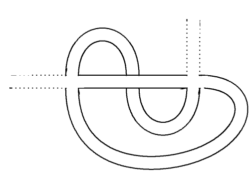

Figure 1: An example of a ribbon graph.

2.3 The 2-dimensional Grosse-Wulkenhaar Model

The 2-dimensional Grosse-Wulkenhaar model or GW_2 model is defined by the Wick-ordered action:

$$\begin{aligned} S &= \int d^2x \big[\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{\mu^2}{2}\phi\star\phi \\ &+ \ :\frac{\lambda}{4}\phi\star\phi\star\phi\star\phi:\big]. \end{aligned} \tag{9}$$

where $\phi(x)$ is a real function defined over the noncommutative coordinates x_μ, $\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu}x^\nu$ and the Euclidean signature has been used. Remark that this model is not translation invariant due to the harmonic oscillator term.

The action (9) has a remarkable symmetry such that if we exchange the position and momentum operator:

$$p_\mu \leftrightarrow \tilde{x}_\mu, \quad \hat{\phi}(p) \leftrightarrow \pi\sqrt{|\det\theta|}\phi(x), \tag{10}$$

where

$$\hat{\phi}(p) := \int d^2x e^{-ip\Theta x}\phi(x), \quad p\Theta x := p_\mu\Theta^{\mu\nu}x_\nu, \tag{11}$$

the action (9) is invariant up to a scalar factor:

$$S[\phi;\mu,\lambda,\Omega] \mapsto \Omega^2 S[\phi;\frac{\mu}{\Omega},\frac{\lambda}{\Omega},\frac{1}{\Omega}]. \tag{12}$$

This symmetry is called the Langman-Szabo duality [23]. It is essential for solving the UV/IR mixing problem hence for proving the renormalisabiliy of the Grosse-Wulkenhaar model.

The renormalisability of the GW model has been studied extensively in the matrix representation [7, 8], the direct space representation [34] and the parametric representation [35]. In this paper we shall use the matrix representation.

In the matrix basis the GW_2 action could be written as:

$$\begin{aligned} S[\phi] &= 2\pi\theta\mathrm{Tr}\Big[\frac{1}{2}\phi\Delta\phi + :\frac{\lambda}{4}\phi^4_\star:\Big] \\ &= 2\pi\theta\sum_{m,n,k,l}\Big[\frac{1}{2}\phi_{mn}\Delta_{mn;kl}\phi_{kl} + \frac{\lambda}{4}:\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}:\Big]. \end{aligned} \tag{13}$$

where ϕ_{mn} is a Hermitian matrix. In this basis the Laplacian reads:

$$\Delta_{mn,kl} = [\mu^2 + \frac{2}{\theta}(m+n+1)]\delta_{ml}\delta_{nk} - \frac{2}{\theta}(1-\Omega^2) \tag{14}$$

$$\times \ \big[\sqrt{(m+1)(n+1)}\delta_{m+1,l}\delta_{n+1,k} + \sqrt{mn}\delta_{m-1,l}\delta_{n-1,k}\big]. \tag{15}$$

We introduce also an ultraviolet cutoff Λ for the matrix indices such that $0 \leq m,n \leq \Lambda$, then the covariance matrix $C_{mn,kl}$ is a $\Lambda^2\times\Lambda^2$ dimensional matrix in general. We could obtain the covariance $C_{sr,kl}$ by the following relation [6]:

$$\sum_{r,s\in\Lambda}\Delta_{mn,rs}C_{sr,kl} = \delta_{ml}\delta_{nk}. \tag{16}$$

When $\Omega = 1$, the kinetic matrix reduces to a much simpler form:

$$\Delta_{mn,kl} = \Big[\mu^2 + \frac{4}{\theta}(m{+}n{+}1)\Big]\delta_{ml}\delta_{nk}, \tag{17}$$

which is a diagonal matrix in the double indices and the covariance matrix $C_{mn,kl}$ reads:

$$C_{mn,kl} = \frac{1}{\mu^2 + \frac{4}{\theta}(m{+}n{+}1)}\delta_{ml}\delta_{nk}. \tag{18}$$

Since $\Omega = 1$ is a fixed point of this theory [25], we shall for simplicity take $\Omega = 1$ in the rest of this paper and write the covariance as $C_{mn} = C_{mn,nm} = \frac{1}{m+n+1}$ for simplicity.

The Wick ordering of the interaction with ultraviolet cutoff Λ reads:

$$:\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} := \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} - 8\phi_{mp}\phi_{pm}T^{\Lambda}_m + 6\mathrm{Tr}_m(T^{\Lambda})^2_m. \tag{19}$$

where

$$T^{\Lambda}_m = \sum_{q=0}^{\Lambda}\frac{1}{q+m} = \log\frac{\Lambda+m}{m} \sim \log\Lambda, \text{for } 1 \leqslant m \ll \Lambda, \tag{20}$$

and

$$T^2_{\Lambda} = \mathrm{Tr}_m(T^{\Lambda}_m)^2 = \sum_m(\sum_p\frac{1}{m+p})(\sum_q\frac{1}{m+q}) \sim (2\ln^2 2 + \frac{\pi^2}{6})\Lambda. \tag{21}$$

The counter terms $\mathrm{Tr}\phi^2 T$ and T^2_{Λ} and an explanation of the combinatorial factors are shown in Figures 2 and 3.

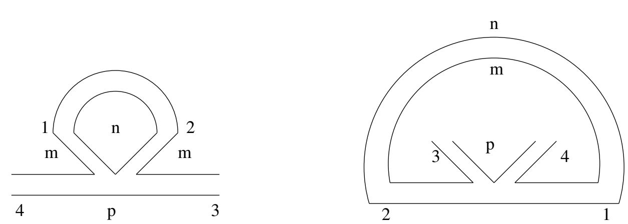

Figure 2: The counter-term $\mathrm{Tr}(\phi^2 T)$ from the Wick ordering. There are 4 possibilities of contracting the nearest neighbor fields, for example, 1 with 2, 2 with 3 , 3 with 4 or 4 with 1; And each contraction has additional two possibilities of whether the two uncontrcted fields are in the external face of the inner face, as shown in this Figure. In total this gives a factor 8.

Note that since the infinite volume limit is hidden into the matrix basis, we don't need any more the cluster expansion. In this sense the Grosse-Wulkenhaar model is even simpler than the commutative ϕ^4_2 theory.

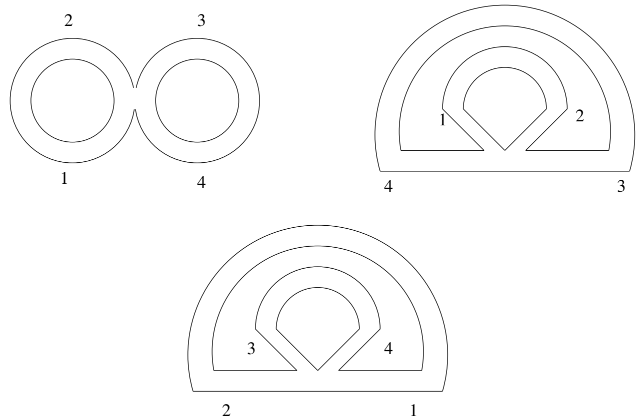

Figure 3: The counter term T_Λ^2 and combinatorics. There are 2 possibilities of forming a vacuum graph, for example 1 contracts with 2 and 3 contracts with 4, or 1 contracts with 4 and 2 contracts with 3. For each case there are an additional 3 possibilities shown in this Figure. In total this gives a factor 6.

3 The intermediate field representation and the Loop vertex expansion

3.1 The intermediate field representation

The partition function for the matrix model reads:

$$Z(\lambda) = \int d\mu(\phi_{mn}) e^{-S[\phi_{mn}]}, \tag{22}$$

where

$$d\mu(\phi_{mn}) = \pi^{-N^2/2} e^{-1/2[\sum_{mn} \phi_{mn}\Delta_{mn}\phi_{nm}]} \prod_{mn} d\mathrm{Re}(\phi_{mm}) d\mathrm{Im}(\phi_{nm}). \tag{23}$$

is the normalized Gaussian measure of the matrix fields ϕ_{mn} with covariance C given by (18) and $S[\phi_{mn}]$ is the Wick ordered interaction term.

We introduce the Hermitian matrix σ_{mn} as an intermediate field and the partition function can be written as:

$$Z(\lambda) = \int d\mu(\sigma) d\mu(\phi) e^{-\frac{1}{2}\sum_{mnkl}\phi_{mn}\Delta_{mn,kl}\phi_{kl} - i\sqrt{2\lambda}(\sum_{kmn}\sigma_{km}\phi_{mn}\phi_{nk} - 4\sum_{mk}\sigma_{km}\delta_{mk}T^\Lambda_m) + \frac{5}{2}\lambda T^2_\Lambda}. \tag{24}$$

where

$$d\mu(\sigma_{mn}) = \pi^{-N^2/2} e^{-1/2\sum_{mn}\sigma^2_{mn}} \prod_{mn} d\mathrm{Re}(\sigma_{mn}) \mathrm{Im}(\sigma_{mn}). \tag{25}$$

is the normalized Gaussian measure for the hermitian matrices σ with the covariance:

$$< \sigma_{mn}, \sigma_{kl} >= \int d\mu(\sigma)\sigma_{mn}\sigma_{kl} = \delta_{nk}\delta_{ml}. \tag{26}$$

Integrating out the matrix fields ϕ_{mn} we have:

$$Z(\lambda) = \int d\mu(\sigma) e^{2i\sqrt{2\lambda}\mathrm{Tr}T^\Lambda\sigma - \frac{1}{2}\mathrm{Tr}\log[1 + i\sqrt{2\lambda}\ C^{1/2}(I\otimes\sigma + \sigma\otimes I)C^{1/2}\ ] + \frac{5}{2}\lambda T^2_\Lambda}, \tag{27}$$

where $\mathrm{Tr} T^\Lambda \sigma = \sum_m T^\Lambda_m$, I is the $\Lambda \times \Lambda$ dimensional identity matrix. It is useful to keep all the indices of the covariance $C = \{C_{mn,kl}\}$ and write is as a $\Lambda^2 \times \Lambda^2$ dimensional diagonal matrix with elements $C_{mn,kl} = \frac{1}{m+n+1}\delta_{ml} \otimes \delta_{nk}$. $C^{1/2}$ is the square root of the covariance C with elements:

$$[C^{1/2}]_{mn,kl} = \sqrt{\frac{1}{m+n+1}}\delta_{ml} \otimes \delta_{nk}. \tag{28}$$

The term $\mathrm{Tr}\log[1 + i\sqrt{2\lambda}\ C^{1/2}(I \otimes \sigma + \sigma \otimes I)C^{1/2}\ ]$ is called the loop vertex, which was first introduced in [27]. Remark that the term $C^{1/2}[I \otimes \sigma + \sigma \otimes I]C^{1/2}$ in the loop vertex is Hermitian. Graphically this term means that the intermediate field σ can attach to either the inner border or the outer border of a ribbon graph.

We define the $\Lambda^2 \times \Lambda^2$ dimensional matrix $\hat{\sigma}$ as

$$\hat{\sigma} = I \otimes \sigma + \sigma \otimes I, \tag{29}$$

then the loop vertex reads: $\mathrm{Tr}\log[1 + i\sqrt{2\lambda}\ C^{1/2}\hat{\sigma}C^{1/2}\ ]$. In the rest of this paper we shall write $\hat{\sigma}$ as σ for simplicity and the loop vertex reads: $\mathrm{Tr}\log[1 + i\sqrt{2\lambda}\ C^{1/2}\sigma C^{1/2}\ ]$. We just need to remember that $C^{1/2}$ and σ are $\Lambda^2 \times \Lambda^2$ dimensional matrix.

Before using the BKAR tree formula to calculate the connected function, we first of all expand the loop vertex term as:

$$\begin{aligned} & -\frac{1}{2}\mathrm{Tr}\log[1 + i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}] \\ = & -\frac{i}{2}\mathrm{Tr}(\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}) - \frac{1}{2}\mathrm{Tr}\log_2[1 + i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}], \end{aligned} \tag{30}$$

where $\log_n(x)$ for $n \geq 2$ is defined as the n-th Taylor remainder of the function $\log(x)$:

$$\log_n(x) = \log(x) - [x - x^2/2 + x^3/3 \cdots + (-1)^{n+1}x^n/n]. \tag{31}$$

The first term in formula (30) reads:

$$-\frac{i}{2}\mathrm{Tr}(\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}) = -2 \cdot \frac{i}{2}\sqrt{2\lambda}\sum_m^\Lambda T^\Lambda_m \sigma_{mm}, \tag{32}$$

where the additional factor 2 is due to that there are two kinds of tadpoles (see Figure 4) when taking the $Trace$ after using the fusion rule for the covariances $C^{1/2}$. Hence the leaf

Figure 4: Tadpoles.

tadpole term (32) could cancel partially the linear counter term of the Wick ordering, see formula (27).

So we could define the new interaction vertex as

$$
\begin{aligned}
V(\sigma) &= -\frac{1}{2}\mathrm{Tr}\log_2[1+i\sqrt{2\lambda}C^{1/2}\hat{\sigma}C^{1/2}] \\
&+ i\sqrt{2\lambda}\mathrm{Tr}T^{\Lambda}\hat{\sigma}+\frac{5}{2}\lambda T_{\Lambda}^2,
\end{aligned}
\tag{33}
$$

where the leaf tadpoles have been already canceled by the counter term and the partition function could be written as:

$$
Z(\lambda)=\int d\mu(\hat{\sigma})e^{V(\hat{\sigma})}. \tag{34}
$$

3.2 The BKAR Tree formula and the expansion

The most interesting quantities in our model are the connected Schwinger's functions. We shall derive them by the BKAR tree formula.

Let us first of all expand the exponential as $\sum_n \frac{V(\hat{\sigma})^n}{n!}$. To compute the connected function while avoiding an additional factor $n!$, we give a kind of $fictitious$ index v, $v=1,\cdots,n$ to each field σ in vertex $V(\sigma)$ and we could rewrite the expanded interaction term as $\sum_n \prod_{v=1}^n \frac{V(\sigma^v)}{n!}$. This means that we consider n different copies σ_v of σ with a degenerate Gaussian measure

$$
d\nu(\{\sigma_v\})=d\nu(\sigma_{v_0})\prod_{v'\neq v_0}^{n}\delta(\sigma'_v-\sigma_{v_0})d\sigma_{v'}\ , \tag{35}
$$

where v_0 is an arbitrarily marked vertex.

The vacuum Schwinger's function is given by:

Theorem 3.1 (Loop Vertex Expansion [27]).

$$
\begin{aligned}
&\log Z(\lambda,\Lambda,\mathcal{V})=\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\mathcal{T}\text{ with } n \text{ vertices}} G_{\mathcal{T}} \qquad (36)\\
&G_{\mathcal{T}}=\Big\{\prod_{\ell\in\mathcal{T}}\sum_{m_\ell,n_\ell,p_\ell,q_\ell}[\int_0^1 dw_\ell]\Big\}\int d\nu_{\mathcal{T}}(\{\sigma^v\},\{w\})\\
&\Big\{\prod_{\ell\in\mathcal{T}}[\delta_{n_\ell p_\ell}\delta_{m_\ell q_\ell}\frac{\delta}{\delta\sigma^{v(\ell)}_{m_\ell,n_\ell}}\frac{\delta}{\delta\sigma^{v'(\ell)}_{p_\ell,q_\ell}}]\Big\}\prod_{v=1}^n V_v,
\end{aligned}
$$

where

- *each line ℓ of the tree joins two different loop vertices $V^{v(\ell)}$ and $V^{v'(\ell)}$ which are identified through the function $\delta_{m_\ell q'_\ell}\delta_{n_\ell p'_\ell}$, since the propagator of σ is ultra-local.*

- *the sum is over spanning trees joining all n loop vertices. These trees have therefore $n-1$ lines, corresponding to σ propagators.*

- *the normalized Gaussian measure $d\nu_{\mathcal{T}}(\{\sigma_v\},\{w\})$ over the fields σ_v has now a covariance*

$$< \sigma^v_{mn}, \sigma^{v'}_{kl} >= \delta_{nk}\delta_{ml} w^{\mathcal{T}}(v,v',\{w\}), \tag{37}$$

which depend on the "fictitious" indices. Here $w^{\mathcal{T}}(v,v',\{w\})$ equals to 1 if $v=v'$, and equals to the infimum of the w_ℓ for ℓ running over the unique path from v to v' in $\mathcal{T}$ if $v\neq v'$.

The proof is a standard consequence of the BKAR formula (see [27, 29]).

The n-point Euclidean Green's functions or the Schwinger's functions, with $n=2p$ even, could be derived similarly by introducing the (matrix) source fields $j=[j_{mn}]$. The partition function with source fields reads:

$$Z(\Lambda, j(x)) = \int d\mu(\phi) e^{-\frac{\lambda}{4}\mathrm{Tr}[(\phi^2-4T_\Lambda)^2+10T_\Lambda^2]+\mathrm{Tr} j\phi} \tag{38}$$

$$= \int d\nu(\sigma) e^{\frac{5}{2}\lambda T_\Lambda^2+\mathrm{Tr}\left(2i\sqrt{2\lambda}T_\Lambda\sigma-\frac{1}{2}\log(1+i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2})\right)+\frac{1}{2}\mathrm{Tr}[jC^{1/2}RC^{1/2}j]},$$

where

$$R = \frac{1}{1+i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}} \tag{39}$$

is called the resolvent matrix. We define also the full resolvent matrix:

$$\hat{R}(\sigma) = C^{1/2}\frac{1}{1+i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}}C^{1/2}. \tag{40}$$

The connected n-point function is given by:

$$S(l_1r_1,\cdots,l_{2p}r_{2p}) = \frac{1}{(2p)}\frac{\partial^{2p}}{\partial j_{l_1r_1}\cdots\partial j_{l_{2p}r_{2p}}}\log Z(\Lambda,j)|_{j=0}. \tag{41}$$

More precisely, we have the following theorem for the connected Schwinger's function:

Theorem 3.2.

$$S^c = \sum_\pi\sum_{n=0}^\infty \frac{1}{n!}\sum_{\mathcal{T}\text{ with } n \text{ vertices}}\Big\{\prod_{\ell\in\mathcal{T}}\Big[\int_0^1 dw_\ell\Big]\Big\}$$
$$\int d\nu_{\mathcal{T}}(\{\sigma^v\},\{w\})\Big\{\prod_{\ell\in\mathcal{T}}\Big[\delta_{m_\ell q_\ell}\delta_{n_\ell p_\ell}\Big]\frac{\delta}{\delta\sigma^{v(\ell)}_{mn}}\frac{\delta}{\delta\sigma^{v'(\ell)}_{pq}}\Big]\Big\}$$
$$\prod_{v=1}^n V_v\prod_{k=1}^{2p}\hat{R}_{l_{1\pi(k)},r_{1,\pi(k)}}(\sigma), \tag{42}$$

where the sum over π runs over the parings of the $2p$ external variables according to the cyclic order and $\hat{R}_{l_{1\pi(k)},r_{1,\pi(k)}}(\sigma)$ are the resolvent matrices whose explicit form are given by formula (39).

So in this paper we consider only the vacuum connected Schwinger's function, as the n-point function could be easily obtained by derivation w.r.t. the source fields.

4 The graph representations and the amplitudes of the LVE

4.1 Direct representation of the LVE

Now we consider the basic graph structure of the Loop vertex expansion, which we call the *direct graphs* or the LVE in the direct representation.

There are three basic line structures or propagators in the LVE:

- The full resolvent $\hat{R}(\lambda, \sigma)$ (see also (40)) is defined as:

$$\begin{aligned}\hat{R}_{mn}(\sigma,\lambda) &= \frac{\partial}{\partial \sigma_{mn}}[-\frac{1}{2}\mathrm{Tr}\log(1+i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2})] \\ &= -\frac{1}{2}i\sqrt{2\lambda}\big[C^{1/2}\frac{1}{1+i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}}\ C^{1/2}\big]_{mn}\end{aligned} \tag{43}$$

  $\hat{R}$ can be written in terms of the resolvent matrix defined in (39) as:

$$\hat{R}_{mn} = [C^{1/2}\ R\ C^{1/2}]_{mn}. \tag{44}$$

- The propagators C_{mn} between the original fields ϕ_{mn},
- The propagators between the σ fields.

The propagators are shown in Figure 5.

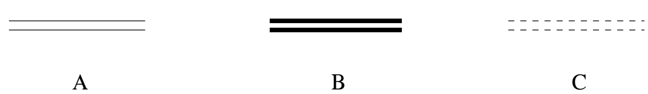

Figure 5: The propagators in LVE. A stands for the resolvent R_{mn}, B stands for the pure propagator C_{mn}, and C is the propagator of the σ fields.

There are three kinds of interaction vertices in the LVE: the counter terms, the leaf terms K with coordination number 1 and the general interaction vertices V. A leaf vertex K is generated by deriving once w.r.t the σ field on the $\log_2$ term:

$$\begin{aligned}K_{mn} &= \frac{\partial}{\partial \sigma_{mn}}[-\frac{1}{2}\mathrm{Tr}\log_2(1+i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2})] \\ &= -\frac{1}{2}i\sqrt{2\lambda}[C^{1/2}(\frac{1}{1+i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}}-1)C^{1/2}]_{mn}.\end{aligned} \tag{45}$$

A general loop vertex could be obtained by deriving twice or more with respect to the σ fields:

$$\begin{aligned}V_{m_1 m_2\cdots m_{p-1}m_p}(\lambda,\sigma) &= \frac{\partial}{\partial \sigma_{m_1 m_2}}\cdots\frac{\partial}{\partial \sigma_{m_{p-1}m_p}}[-\frac{1}{2}\mathrm{Tr}\log(1+i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2})] \\ &= -\frac{1}{2}(i\sqrt{2\lambda})^p(-1)^p\sum_{\tau}\hat{R}_{m_1 m_{\tau(1)}}(\sigma,\lambda)\cdots\hat{R}_{m_{\tau(p)}m_1}(\sigma,\lambda).\end{aligned} \tag{46}$$

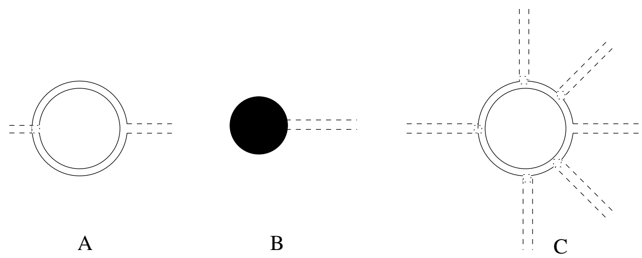

Figure 6: The basic graph elements of LVE. Graph A means the leaf K, graph B means the counter term and graph C means the most general loop vertex which has several σ fields attached.

with $p \geq 2$ and the sum over τ is over the p cyclic permutations of the resolvents.

The basic interaction vertices are shown in Figure 33, where we didn't show explicitly the pure propagator $C^{1/2}$.

After integrating out the σ fields we obtain the vacuum graphs which are planar ribbon graphs without broken faces. The first order graphs whose amplitudes are proportional to λ are shown in Figure 7. Graph C in this figure means the counter term $\frac{5}{2}\lambda T_\Lambda^2$ that appears only in the first order expansion.

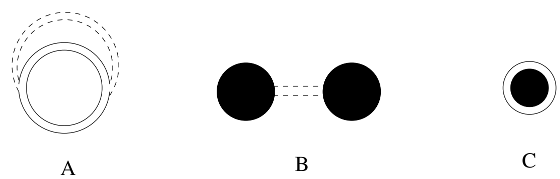

Figure 7: The graphs of first order expansion. We have taken account of all possible configurations of graph A in the calculation.

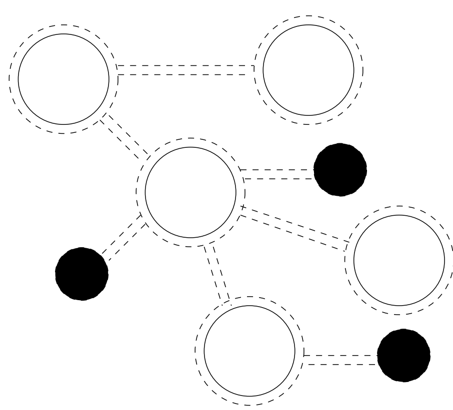

Figure 8: A general graph of Loop vertex expansion. Here the leaf means the leaf loop vertex K.

Since we consider a real symmetric matrix model, the leaf loop vertices and the counter terms could appear either in the outer side, in the inner side or both sides of a loop vertex, we call the graph of the first case the normal direct tree and the last two case the inner direct

tree. The normal direct tree and the inner direct tree are shown in Figure 8 and Figure 9, respectively. [1]

The amplitude of the two graphs are exactly the same, so in this paper we only consider explicitly the first case for the direct graph representations and the dual representations. The contributions of the inner direct trees to the amplitude should be taken into account by calculating correctly the corresponding combinatorial factors. A general graph of LVE is

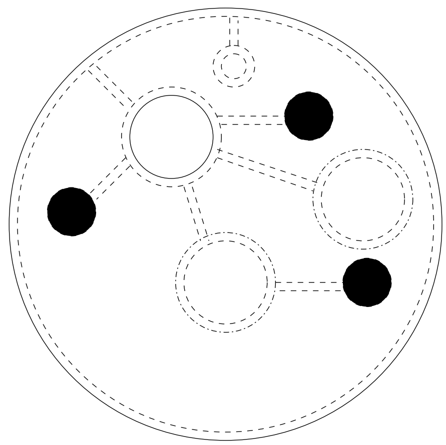

Figure 9: A tree of loop vertices where the counter terms or leaf terms are attached to the inner part of the loop vertices. Here the leaf means the leaf loop vertex K.

shown in Figure 8 and Figure 9.

Now we have the following definitions for the objects in $\mathcal{T}$: To each counterterm $\mathrm{Tr}\sigma T_\Lambda D_{mn}$, where D_{mn} is the pure propagator generated by the derivation $\frac{\partial}{\partial\sigma_{mn}}$ on the loop vertex, we associate a tree line index ℓ_c; to each leaf term $K_{mn}(\sigma)$ we associate a tree line index ℓ and to each full resolvent $R(\sigma)_{mn}$ sandwiched by the two tree lines ℓ_l and ℓ_r, such that ℓ_l is on the left side and ℓ_r is on the right side, we associate two tree line indices ℓ_l and ℓ_r .

So the amplitude for a tree of loop vertices reads:

$$G_{\mathcal{T}} = \int d\mu(\sigma)\mathrm{Tr}[\prod_{\ell\in\mathcal{T}} K^{\ell}_{mn}(\sigma) \prod_{\ell_c\in\mathcal{T}} (\sqrt{2\lambda}T_\Lambda D^{\ell_c}_{mn}) \prod_{\ell_l,\ell_r\in\mathcal{T}} \hat{R}^{\ell_l.\ell_r}_{mn}(\sigma)], \tag{47}$$

where the global Tr means that we need to take the cyclic order of all the objects. The indices ℓ_l or ℓ_r may coincide with the indices ℓ or ℓ_c.

4.2 The dual representation

Since a LVE graph in the direct representation is *planar*, the notion of duality is globally well-defined. In the dual representation we have a canonical (up to an orientation choice) and more explicit cyclic ordering of all ingredients occurring in the expansion [32].

[1] Remark that for the orientable model with interaction $\lambda\bar{\phi}\phi\bar{\phi}\phi$ the loop vertex has the form $\mathrm{Tr}\log(C\sigma_L\otimes 1+1\otimes 1)$ or $\mathrm{Tr}\log(1\otimes 1+1\otimes\sigma_R C)$, which means that the σ fields appear only on one side of the loop vertex (left or right). See also [27] for a more detailed discussion on this point .

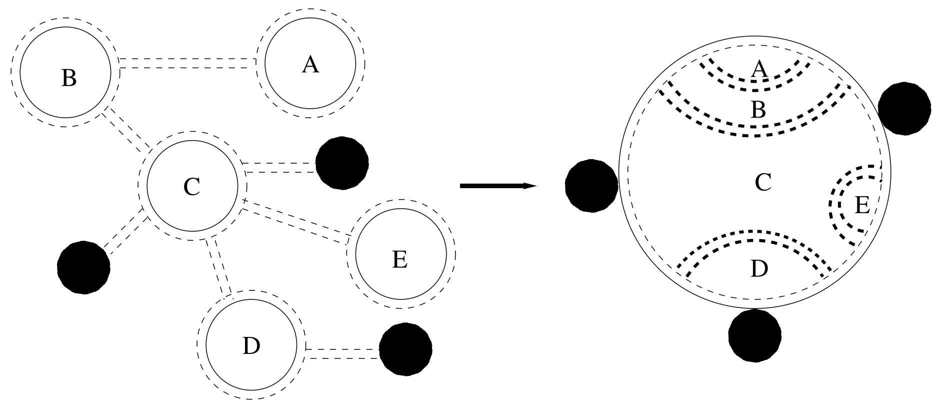

Figure 10: The dual graph of a LVE. The area enclosed by the bold dash ribbons correspond to the original loop vertices.

. In the dual representation the original tree of loop vertices $\mathcal{T}$ is replaced by the dual tree $\bar{\mathcal{T}}$, which consists of a single huge loop vertex $\mathcal{C}$ with dual tree line $\bar{\ell}$. Each dual tree line $\bar{\ell} \in \bar{\mathcal{T}}$ corresponds to a line $\ell \in \mathcal{T}$ in the direct picture, and is pictured as a bold dash line. See Figure 10 for a dual tree.

The important fact reflecting the tree character of $\mathcal{T}$ is that these bold dash lines *cannot cross*. Hence they divide this disk $\mathcal{C}$ into different connected regions $\bar{v} \in \bar{V}$, each of which corresponds to a single loop vertex $v \in V$ of the direct expansion. This explains how the two pictures, dual of each other, are equivalent and how anyone can be reconstructed from the other.

This cycle $\mathcal{C}$ contains all the ordinary full lines (ϕ resolvents) of the direct representation and the counter terms, but reread in the cyclic order obtained by turning around the tree. The order of all the objects in the dual representation should be the same as in the direct representation.

Here is the notation for the objects $\mathcal{O} = \bar{B} \cup \bar{S} \cup \bar{\mathcal{T}} \cup \mathcal{L}$ in the dual representation:

- The counter-terms that are pictured as black vertices $\bar{b} \in \bar{B}$. More exactly, each counter-term is considered as a black vertex attached to a pure propagator, so that the amplitude of each counter-term equals to the product of the amplitudes of the two objects. See formula (48). To each counterterm we associate an index $\bar{\ell}_c$.

- The bold dash lines $\bar{\ell} \in \bar{\mathcal{T}}$ that correspond to the original σ propagators. We also call them the dual tree line.

- Simple leaves $\bar{v}_0 \in \bar{S}$, i.e resolvent lines surrounded by a single dual tree line $\bar{\ell}$. For example, the normal lines in region A and E in Figure 10 correspond to two simple leaves. To each leaf we associate an index $\bar{\ell}$.

- Non-leaf resolvent lines $\bar{v}_1 = \{\bar{\ell}_l, \bar{\ell}_r\} \in \mathcal{L}$, ie resolvent lines not in $\bar{S}$ that correspond to the full resolvents $R^{\ell_l,\ell_r}_{mn}(\sigma)$ in the direct representation. We follow the clockwise direction in the dual graph and use two indices $\bar{\ell}_l$ and $\bar{\ell}_r$ to indicate each non-leaf resolvent line sandwiched between the two different dual tree lines $\bar{\ell}_l$ and $\bar{\ell}_r$.

The beauty of the LVE representation is that all the various traces of the loop vertices in the direct representation give rise to a *single* trace in the dual representation. This is the fundamental observation which made the representation suited for constructive matrix madel and non commutative quantum field theories [27].

We need a label u to describe the various objects met when turning around the cycle $\mathcal{C}$. Then to each object $u \in \mathcal{O}$ is associated an operator P_u, with value

$$
\begin{aligned}
P_u &= T_\Lambda D^{\bar{\ell}_c}_{mn} \text{ if } u = \bar{b} \in \bar{B}, &(48)\\
P_u &= K^{\bar{\ell}}_{mn}(\sigma) \text{ if } u = \bar{v}_0 \in \bar{S}, &(49)\\
P_u &= R^{\bar{\ell}_l, \bar{\ell}_r}_{mn}(\sigma) \text{ if } u = \bar{v}_1 \in \mathcal{L}, &(50)
\end{aligned}
$$

In this dual picture the measure $d\nu(\sigma, w)$ corresponds to the following rule: the weakening factor between a $\sigma_{\bar{v}}$ and a $\sigma_{\bar{v}'}$ is the infimum of the w parameters of the lines $\bar{\ell}$ that have to be *crossed* to join the two regions $\bar{v}$ and $\bar{v}'$.

Hence the amplitude for a tree of loop vertices reads:

$$
\begin{aligned}
P_\Lambda = \log Z(\lambda, \Lambda, \mathcal{V}) &= \sum_{\bar{\mathcal{T}} \text{ lines, } |\bar{\mathcal{T}}| = n-1} G_{\bar{\mathcal{T}}} \\
G_{\bar{\mathcal{T}}} = (-\lambda)^{n-1} &\int d\nu(\sigma, w) \mathrm{Tr} \Big\{ \vec{\prod}_{u \in \mathcal{O}} P_u \Big\}. \qquad (51)
\end{aligned}
$$

where the operator Tr means that we sum over all matrix indices following the cyclic ordering of all the objects P_u.

Theorem 4.1. *The vacuum Green's function P_Λ is absolutely convergent and defines an analytic function in the half-disk $\mathcal{D}_\Lambda = \{\lambda \mid \Re(\lambda^{-1}) \geq K \log \Lambda$, where K is a large constant.*

Proof. This is just an application of the techniques of [27, 29] which we recall briefly.

- The number of trees is n^{n-2}, by Cayley's theorem,
- each full resolvent $R(\sigma)_{mn}$ is bounded by 1 in the disk,
- each decorating counter term is bounded by $\log \Lambda$.

So the connected function is bounded by

$$
P_\Lambda \leq \sum_n \frac{n^{n-2}}{n!} \lambda^n (\ln \Lambda)^n \leq \sum (\lambda K)^n (\ln \Lambda)^n. \qquad (52)
$$

which is a convergent geometric series as long as $\Re(\lambda^{-1}) \geq K \log \Lambda$ but divergent when $\Lambda \to \infty$. So the loop vertex expansion alone couldn't tame all divergences and we still need renormalization. Remark that the role of the divergent tadpole terms and counter-terms are in some sense changed. The divergences are coming from the linear counter terms due to the Wick ordering of the interaction and should be compensated by the inner tadpoles, crossings and nesting lines (see page 19 and Figures (13), (16) for the definitions and pictures of these objects) that we generate by contracting the σ fields hidden in the resolvents.

So that we shall put the perturbation series and the *remainders* on the same footing, which is another interesting property of the loop vertex expansion. We shall discuss this point in more detail in the next section. □

5 The Cleaning Expansion

In this section we shall introduce another expansion, which we call the cleaning expansion, to compensate the divergent counter terms. We shall work in the dual representation of the LVE, since in this representation the cyclic ordering of different objects is more explicit. Instead of having a trace operator for every objects, we have a single global trace operator. We shall use the multi-scale representation of the propagators and the resolvents.

We impose also the stopping rule for the cleaning expansion to make sure that we don't expand forever (this would lead to divergence). We stop the expansion until we have gained enough convergent factors to compensate the divergent Nelson's factor.

5.1 The sliced propagator and resolvents

We introduce the Schwinger parameter representation the propagator:

$$C_{mn} = \int_0^\infty d\alpha e^{-\alpha(\frac{\mu^2}{\theta}+m+n+1)} = K\int_0^1 d\alpha e^{-\alpha(\frac{\mu^2}{\theta}+m+n+1)}. \tag{53}$$

We decompose the propagator as:

$$C_{mn} = \sum_{j=0}^{\infty} C^j_{mn}, \tag{54}$$

where

$$C^j_{mn} = \int_{M^{-2i}}^{M^{-(2i-2)}} d\alpha e^{-\alpha(\mu^2+\frac{4}{\theta}(m+n+1))}, \tag{55}$$

and M is an arbitrary positive constant. We could easily find that

$$|C^j_{mn}| \leq KM^{-2j}e^{-M^{-2j}||\mu^2+\frac{4}{\theta}(m+n+1)||}. \tag{56}$$

We have the associated decomposition for the amplitude of the connected function:

$$G_T = \sum_j G^j_T. \tag{57}$$

Due to the cyclic order of the global Trace operator, we could rewrite the loop vertex in the non-symmetric from $\mathrm{Tr}\log[1+i\sqrt{2\lambda}\sigma C]$ and the resolvent defined in formula (39) is written as:

$$R_{mn} = \frac{\partial}{\partial\sigma_{mn}}[-\frac{1}{2}\mathrm{Tr}\log(1+i\sqrt{2\lambda}\sigma C)] = -i\frac{1}{2}\sqrt{2\lambda}\sum_p[\frac{1}{1+i\sqrt{2\lambda}\sigma C}]_{mp}C_{pn}. \tag{58}$$

It is convenient to introduce further notations to write down in more compact form the contribution, or amplitude of a tree in the LVE. We put

$$D_{mn} = i\sqrt{2\lambda}C_{mn}, \tag{59}$$

so we have

$$R(\sigma) = \frac{1}{1+\sigma D}, \text{ and } R(\sigma)_{mn} = [R(\sigma)D]_{mn}, \tag{60}$$

where D_{mn} and $R(\sigma)$ should be thought of as a matrix operator.

Then we define the sliced resolvent as:

$$R^j(\sigma) = \frac{1}{1+\sum_{k\leqslant j} i\sqrt{2\lambda}\ \sigma C^k} = \frac{1}{1+\sum_{k\leqslant j}\ \sigma D^k}, \tag{61}$$

and the full resolvent as

$$\hat{R}^j_{mn} = [D^j\ R^j]_{mn}. \tag{62}$$

For each resolvent we have the algebraic equation

$$\begin{aligned} R^j(\sigma) &= \frac{1}{1+\sum_{k\leq j}\sigma D^k} = \frac{1}{1+\sum_{k<j}\sigma D^k + \sigma D^j} \\ = \quad & R^{j-1}(\sigma)\frac{1}{1+R^{j-1}(\sigma)\sigma D^j} = R^{j-1}(\sigma) - R^{j-1}(\sigma)\sigma D^j R^j. \end{aligned} \tag{63}$$

5.2 The cleaning expansion

In this section we shall expand the resolvents and contract the σ fields to generate the inner tadpoles for canceling (partly) the counter terms and to generate the crossings and nesting lines for gaining convergent factors. This is called the cleaning expansion [32].

The canonical cyclic ordering of the LVE is also essential for the cleaning expansion, as otherwise it would be difficult to find the corresponding "weakening parameters" rule for that Taylor remainder term. But fortunately the cyclic ordering solves nicely this problem! The beginning of the cycle $\mathcal{C}$ is explicitly and fully cleaned of potential tadpoles; the rest of the cycle $\mathcal{C}$ has *no weakening parameters* on the remaining potential tadpoles!

We won't just expand each resolvent naively as power series of $i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}$, since that would generate unnecessarily complicated weakening factors when we contract the σ fields of different regions in dual graph, and what's more, this naive expansion is not convenient for the multi-scale analysis of the amplitude, which is crucial to analyze the Taylor remainder terms.

We apply the cleaning expansion in two steps:

In step 1, we start from an arbitrary marked point in the dual graph and apply the algebraic induction formula (64)

$$R^j(\sigma) = R^{j-1}(\sigma) - R^{j-1}(\sigma)\ \sigma\ D^j R^j \tag{64}$$

which rewrites the multi-scale representation of each resolvent of scale j into two parts, for the resolvent we meet clockwisely. The first part is resolvent of a lower scale $j-1$, and the second part is a product form of a pure propagator of scale j, a resolvent of scale j and a resolvent of scale $j-1$. We start form the highest scale j_{max}. Due to the cyclic ordering the pure propagator term appears only on the left side of the product.

So by using the formula (64) inductively we could lower down the scale of the resolvents and propagators on the one hand and decompose the resolvents into a cleaning part which is the pure propagator with σ field attached and the uncleaned resolvents, on the other hand.

In step 2, we use the integration by parts of the σ fields with respect to the Gaussian measure. For example, for each term in the second part of formula (64) we have:

$$\begin{aligned} & \int d\mu(\sigma) R^{j_{max}-1}(\sigma)\sigma D^j R^{j_{max}}(\sigma) \times \text{ other resolvents of scale } \mathrm{j_{max}} \\ = & \int d\mu(\sigma) \frac{\partial}{\partial \sigma_{nm}} [R^{j_{max}-1}(\sigma) D^{j_{max}} R^{j_{max}}(\sigma) \times \text{ other resolvents}]. \end{aligned} \tag{65}$$

We forget the case where $\frac{\partial}{\partial\sigma}$ acts on the $R^{j_{max}-1}$ term as this gives minor contribution and we consider only the case $\frac{\partial}{\partial\sigma} R^{j_{max}}$ for the first resolvent $R^{j_{max}}$ we meet, for which we have:

$$\begin{aligned} \frac{\partial}{\partial\sigma} D^j R^j &= -D^j \frac{1}{1+\sum_j \sigma D^j} [\sum_j D^j] \frac{1}{1+\sum_j \sigma D^j} \\ &= -D^j \ R^j [\sum_j D^j] \ R^j. \end{aligned} \tag{66}$$

Here we have ignored the matrix indeces. This corresponds to the second line of Figure(11), which shows the typical cleaning expansion process. Then for each resolvent sandwiched by two pure propagator D we use the formula $\frac{1}{1+x} = 1 - x\frac{1}{1+x}$:

$$\begin{aligned} R^j_{mn} &= [\frac{1}{1+\sum_j \sigma D^j}]_{mn} = [1 - \sum_j \sigma D^j \ R^j]_{mn} \\ &= \delta_{mn} - \sum_p [\sum_j \sigma D^j]_{mp} \ R^j_{pn}. \end{aligned} \tag{67}$$

The first term in formula (67) means we identify the two pure propagators generated from the cleaning expansion so that we generate an inner tadpole. Remark that from the above process we don't generate a propagator with a single scale D^j but $\sum_{j=j_{min}}^{j_{max}} D^j$, the tadpole we generated means the sum $T_\Lambda = \sum_j T^j \sim \ln\Lambda$. This corresponds to the left hand side of the third line in Figure (11). In the second term of formula (67) we have again a single σ field in the numerator and we shall perform the integration by parts. There exists still several possibilities:

- If the derivation $\frac{\delta}{\delta\sigma}$ act on the nearest resolvent, we shall repeat the process (67).
- if the derivation act on the second resolvent, this would generate a crossing line.
- it is possible that the σ always contracts with the nearest resolvents without generating a crossing nor an innertadpole. In this case the lines of σ propagators are called nesting lines. See Figure (12) .

We ignore the counter terms which could be anywhere in the dual graph, just for better explanation, as they don't change the cleaning process.

Now we consider the amplitudes for crossing lines and nesting lines. A crossing means a crossing of two σ propagators that correspond to δ functions and are pictured as dash lines, and the σ propagator that overlaps another is attached to a *pure* propagator. Since a

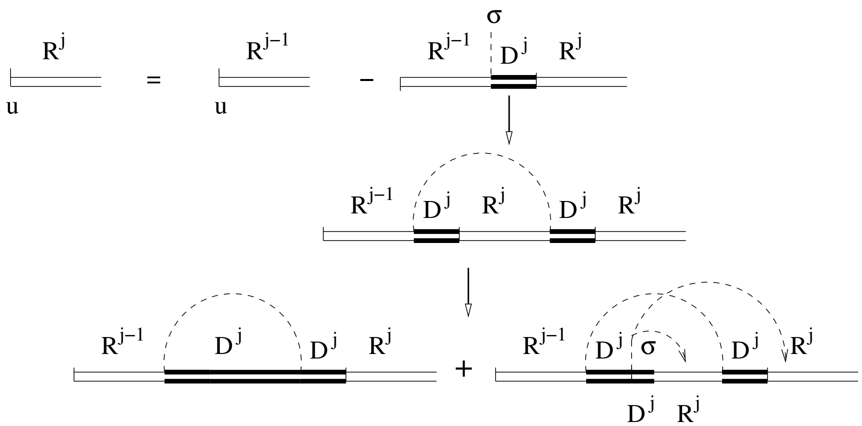

Figure 11: The cleaning expansion. The ordinary ribbon stands for the resolvents and the bold double line stands for the pure propagator. The dashed lines should be envisioned also as double lines and these stand for the σ fields. u is the marked point.

crossing line doesn't form a tadpole, the amplitude of each crossing line equals to the bound of the sliced propagator (see formula (56)):

$$||C^j_{mn}|| \leqslant |KM^{-2j}e^{-M^{-2j}||\mu^2+\frac{4}{\theta}(m+n+1)||}| < KM^{-2j}, \tag{68}$$

Nesting lines means many σ propagators that each one is overlapped by another. Each nesting line is made up of a σ propagator with two pure propagator attached to each end of the σ propagator, and many (at least one) resolvents or /and pure propagators being overlapped by the σ propagator.

The amplitude of a nesting line is bounded by:

$$|R^{\bar\ell_l,\bar\ell_r\,j}_{mn}[\prod R^{\bar\ell'_l,\bar\ell'_r\,j}_{m'n'}]| < |R^{\bar\ell_l,\bar\ell_r\,j}_{mn}C^j_{mn}| \leqslant ||R^j||||C_{mn}|| < KM^{-2j}, \tag{69}$$

where $[\prod R^{\bar\ell'_l,\bar\ell'_r\,j}_{m'n'}]$ means the full resolvents overlapped by the nesting line that we consider and C^j_{mn} means the pure propagator attached to either side of the σ propagator. Since they don't form any closed tadpole, it doesn't matter which C^j_{mn} we consider; we have used the fact that each full resolvent $R^{\bar\ell'_l,\bar\ell'_r\,j}_{m'n'}$ is bounded by 1.

So we gain a convergent factor M^{-2j} for each of the two cases.

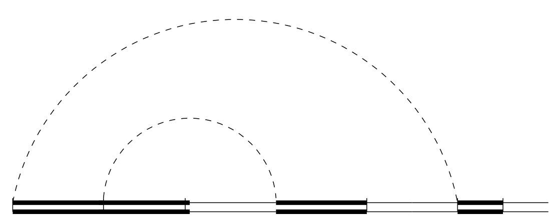

Figure 12: A graph with 2 nesting lines.

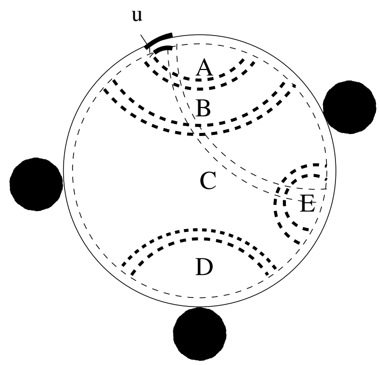

Figure 13: The crossing generated by the case that the field σ_1 is contracted with other resolvents which is not the first one in formula (65), for a more general case; The bold double dash line means the original dual σ propagator, the normal double dash line means the σ propagator generated by the cleaning expansion. Clearly this forms three crossings. u is the marked point.

5.3 The stopping rules

We cannot do the cleaning expansion forever, as this would develop the full perturbation series and ultimately diverge. More exactly, for each scale j the number of crossings shouldn't exceeds $n(j) = (aj)!$, where a is a positive constant to be fixed, otherwise we would no longer be able to compensate for that divergent factorial with the convergent factors coming from the crossings and/or nestings.

So we impose the following stopping rules for the cleaning expansion and write the rest terms as Taylor integral remainder when enough cleaning, depending on the scale, has succeeded:

- We start from resolvents and pure propagators of scale j_{max} and go on the expansion towards lower scales.

- We stop the expansion whenever we generate an inner tadpole and we cancel it with its corresponding counter-term.

- We go on the expansion when we generate a crossing or nesting lines until the total number of crossings and nestling lines reaches $n(j_{max}) = aj_{max}^2$ [2], so that in total we could gain a convergent factor $M^{-aj_{max}^3}$ from the crossings and the nestling lines while lose a divergent combinatorial factor $(aj_{max})! \sim e^{2j_{max}^2 \ln j_{max}}$, which could be easily compensated.

- It is very possible that we couldn't generate enough crossings by expanding a single resolvent in the dual tree. Then we need to go to the next resolvent and repeat the same process.

After the cleaning expansion, no inner tadpole terms nor counter-terms exist in the cleaned part in the dual graph. We shall check this nontrivial fact in section 6. However

[2] The author is very grateful to V. Rivasseau for this suggestion.

the amplitude could still be divergent due to the possible arbitrarily large number of counter terms attached to the uncleaned part of the graph. Instead of further generating inner tadpoles we choose to re-sum them. After the resummation we would generate a divergent factor which we call the Nelson's factor. But this factor is not dangerous since the convergent factors we gained from the crossings and nesting lines would be enough to cancel it. We shall go back to this point in section 7.

6 The renormalization

6.1 The power counting theorem.

Theorem 6.1. *The amplitude of a graph of V vertices, B broken faces and genus g is proportional to*

$$\Lambda^{4-2V-4g-2B} \tag{70}$$

where Λ is the ultraviolet cutoff [5], g is the genus and B is the number of broken faces.

Remark that unlike the commutative ϕ_2^4 model, the power counting for the vacuum Schwinger's functions is different from that for n-point Schwinger's functions due to the number of broken faces.

We could prove the power counting theorem in at least three representations: the matrix base representation [5] [8], the direct space representation and [34] and the parametric representation [35].

We shall not prove this theorem in this paper, as what we need essentially are the topological and combinatorial properties of the ribbon graphs. Since the LVE doesn't change the theory, it doesn't change the power counting either.

6.2 The renormalization

Since the leaf tadpoles could be compensated by the counter terms before we use the LVE, the only possible divergences are the counter terms, and we use the inner tadpoles, the crossings and the nesting lines to compensate them. To be more precise, we use the inner tadpole to compensate the counter terms attached to the cleaned part and use the crossings and the nesting lines to compensate the uncleaned tadpoles.

We have the following theorem for the renormalization, by which we mean the compensation between inner tadpoles and counter terms :

Theorem 6.2. *Each inner tadpole should be canceled by a counter-term.*

The compensation of the uncleaned tadpoles will be discussed in the next section.

Proof. We consider an inner tadpole generated from an arbitrary resolvent $R(\sigma)$. We Taylor expand this resolvent as

$$R(\sigma) = \frac{1}{1 + i\sqrt{2\lambda}C\sigma} = 1 - i\sqrt{2\lambda}C\sigma + (i\sqrt{2\lambda}C\sigma)^2 + \text{Remainder}, \tag{71}$$

and the amplitude for an inner tadpole reads:

$$T_{tadpole} = \int d\nu(\sigma)(\sum_{mn}^{\Lambda} i\sqrt{2\lambda}C_{mn}\sigma_{nm})^2 = -2\lambda T_\Lambda^2. \quad (72)$$

The amplitude for a counter term attached to this resolvent reads:

$$T_c = \int d\nu(\sigma)\frac{1}{2!} \times 2 \times (i\sqrt{2\lambda}\sum_{mn} C_{mn}\sigma_{nm})\mathrm{Tr}_p\sigma_{pp}T_p = 2\lambda T_\Lambda^2. \quad (73)$$

So we have

$$T_{tadpole} + T_c = 0. \quad (74)$$

Remark that since the cancelation is exact, this theorem is true also for the sliced inner tadpoles and counter terms at any scale j. For proving this, we could just use the same method while replacing each propagator C by $\sum_j C^j$ and each counter term T_Λ by $\sum_j T_j$.

So we have proved the remormalization theorem. We could also use the combinatorial renormalization (see the appendix) to prove the cancelation of the inner tadpoles with the counter terms. We don't discuss this point in detail in this paper as it is not essential. The interested reader could also look at [32] and [31] for more details for the combinatorial properties for the LVE. □

7 Nelson's argument and the bound of the connected function

In the last section we introduced the cleaning expansion, after which all inner tadpoles should be canceled by the counter-terms in the cleaned part of the dual graph. But there might still be arbitrarily many counter terms in the uncleaned part and they are divergent. Instead of canceling all of them, we resum them by using the inverse formula of the Gaussian integral and integration by parts.

We write more explicitly, but loosely, the amplitude of the connected function after the LVE and the cleaning expansion as:

$$A_T^N = \prod_{\bar\ell,\bar\ell_c\bar\ell_l,\bar\ell_r\in\mathcal{T}} \int d\mu(w,\sigma)\int_0^1 dw_l \mathrm{Tr}\overrightarrow{\prod}[K_{mn}^{\bar\ell}(\sigma)R_{mn}^{\bar\ell_l,\bar\ell_r}(\sigma)(D^{\bar\ell_c}T_\Lambda)], \quad (75)$$

where $R_{mn}(\sigma)$ is the full resolvent that contains the pure propagator C, Tr needs to follow the cyclic order according to the real positions of the leaf terms K, the resolvents R and the counter terms $T_\Lambda D$. We have used the fact that all weakening factors for the counter terms equal to one, as they are leaves in the graph. There are only weakening factors for the σ propagators that cross different regions in the dual graph.

Now we consider the function G for which we haven't expanded the counter-term:

$$G = \int d\nu(\sigma,w)\prod_{l\in\mathcal{T}} \mathrm{Tr}[K_{mn}R_{mn}(\sigma)]e^{\mathrm{Tr}i\sqrt{2\lambda}\sigma T_\Lambda}. \quad (76)$$

We use the formula

$$\int d\nu(w,\sigma) f(\sigma) g(\sigma) = e^{\frac{1}{2}\frac{\partial}{\partial\sigma}C(\sigma,\sigma',w)\frac{\partial}{\partial\sigma}} f(\sigma)g(\sigma))|_{\sigma=0}, \tag{77}$$

where $C(\sigma,\sigma',w)$ is the covariance that might depend on the weakening factor w or not. Hence

$$\begin{aligned} G &= \int d\nu(\sigma,w) \sum_{N=0}^{\infty} \frac{1}{N!}[\frac{1}{2}\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}]^N \{ \prod_{\bar{l}\in\bar{\mathcal{T}}} \int_0^1 dw_{l'} \mathrm{Tr}[K^{\bar{\ell}}(\sigma) R^{\bar{\ell}_l,\bar{\ell}_r}(\sigma)] e^{\mathrm{Tr} i\sqrt{2\lambda}\sigma T_\Lambda}]\} \\ &= \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\sum_{N_3=0}^{\infty}\prod_{l\in\mathcal{T}}\int_0^1 dw_l \frac{1}{N_1! N_2!}(\frac{1}{2})^{N_1+N_2}[(\frac{\partial}{\partial\sigma})^{N_2}\{\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}\}^{N_1}][K(\sigma)R(\sigma)] \\ &\quad \{(\frac{\partial}{\partial\sigma'})^{N_2}\frac{1}{N_3!}[\frac{1}{2}\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}]^{N_3} e^{\sum_q i\sqrt{2\lambda}\sigma_{mm}T_m^\Lambda}\}|_{\sigma=0}. \end{aligned} \tag{78}$$

where we have written down expicitly the trace term in the exponential, and the term T_m^Λ is defined by formula (20).

While the N_1 and N_2 derivations generate connected terms, the last derivatives generate N_3 disconnected terms, see graph B in Figure 6.

If we sum the the indices m for the counterterms directly, we would have:

$$\begin{aligned} \sum_{N_3=0}^{\infty}\frac{1}{N_3!}[\frac{1}{2}\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}]^{N_3} e^{\mathrm{Tr} i\sqrt{2\lambda}\sigma T_\Lambda}|_{\sigma=0} &= \sum_{N_3=0}^{\infty}\frac{1}{N_3!}[\frac{1}{2}\sum_{m=0}^{\Lambda}(i\sqrt{2\lambda}T_m^\Lambda)^2]^{N_3} \\ = e^{-\lambda T_\Lambda^2}., \end{aligned} \tag{79}$$

And the resumed amplitude reads:

$$A_T^N = \int d\nu(\sigma,w) \prod_{\bar{\ell}\in\bar{\mathcal{T}}} \mathrm{Tr} \overrightarrow{\prod} [K_{mn}^{\bar{\ell}}(\sigma) R_{mn}^{\bar{\ell}_l,\bar{\ell}_r}(\sigma)] e^{\mathrm{Tr} i\sqrt{2\lambda}\sigma T_\Lambda} e^{\lambda T_\Lambda^2}, \tag{80}$$

We find that by resuming the counterterms we need to pay for a divergent factor $e^{\lambda T_\Lambda^2}$, which is also called Nelson's factor. But as is shown in Formula (21), $T_\Lambda^2 \sim \Lambda$ but not $\ln^2\Lambda$ when $\Lambda\to\infty$, so that this factor couldn't be compensated by the convergent factor $e^{-a(\ln\Lambda)^3}$ that we have gained.

For solving this problem we need to analyze more carefully the scale indices of the counter terms and other crossings before summing them [3].

7.1 Improved Nelson's factor

The reason for the bad factor Λ is due to that we sum all indices in $\sum_{m=0}^{\Lambda}(i\sqrt{2\lambda}T_m^\Lambda)^2$ in formula (79), which means we detach all counter terms in the uncleaned part of the dual graph even if they are convergent. For example, the tadpole of amplitude $\ln(\Lambda/m)$, where m is the scale of the border of the ribbon where the tadpole is attached, is not divergent if

[3]The author is very grateful to V. Rivasseau for the discussion about this point.

$m \sim c\Lambda$, where c is a number that is not much smaller than 1, say 1/2. What's more, we need to bear in mind that each counterterm is attached to a pure propagator in the form of $\lambda C_{nm} T_m^\Lambda$ and C_{nm} decays as M^{-2j}, where $2j \sim \ln\max(m,n)$. Here we have ignore all the inessential factors.

So instead of considering only T_m^Λ, we need to take the whole object $C_{nm}T_m^\Lambda$, which we also call it the full counter term, into account. We first consider the case $m > n$, so we have $m \sim M^{2j}$ where $M > 1$ is an arbitrary constant. Then we have

$$C_{nm}T_m^\Lambda \sim \frac{1}{m}\ln(\Lambda/m) \leq \frac{\ln\Lambda}{m}. \tag{81}$$

So that the counterterm doesn't cause any divergence as long as $m > \ln\Lambda$ and we could just bound these counter terms by 1 and need not to detach them from the connected graph. The counter terms become dangerous only when $m \leq \ln\Lambda$ and we need to resumm them as introduced before.

So that we only need to sum over the indices m for $m \leq \ln\Lambda$ in formula (79), which reads:

$$-\quad 2\lambda\sum_{m=1}^{\ln\Lambda}\ln^2(\Lambda/m) < -2\lambda\sum_{m=1}^{\ln\Lambda}\ln^2\Lambda = -2\lambda(\ln\Lambda)^3, \tag{82}$$

and the amplitude of the connected function after the resummation reads:

$$A_T^N = \int d\nu(\sigma,w)\prod_{\bar\ell\in\bar{\mathcal{T}}}\ \mathrm{Tr}\overrightarrow{\prod}[K_{mn}^{\bar\ell}(\sigma)R_{mn}^{\bar\ell_l,\bar\ell_r}(\sigma)]e^{\mathrm{Tr}i\sqrt{2\lambda}\sigma T_\Lambda}e^{\lambda(\ln\Lambda)^3}, \tag{83}$$

whose divergent behavior is much better than in (80).

If $m \leq n$ and $n \leq \ln\lambda$, we still need to resum the counter terms but the divergent factor should be smaller. If $n > \ln\Lambda$, then the corresponding tadpole is convergent and we could just bound it by 1.

This resummation process is shown in Figure 14.

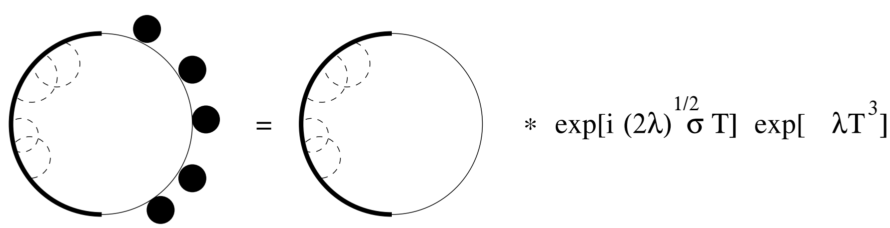

Figure 14: A sketch of the resummation of the counter terms. Where the term T means $\ln\Lambda$. The counterterms with the scale index of outer border larger than $\ln\ln\Lambda$ are not summed.

The divergent factor $e^{\lambda(\ln\Lambda)^3}$ is not dangerous as we could bound it by the following formula:

$$e^{-aj_{max}^3}\cdot(j_{max}^2)!\cdot e^{\lambda\ln^3\Lambda} \sim e^{-aj_{max}^3+2j_{max}^2\ln j_{max}+\lambda j_{max}^3} < 1 \tag{84}$$

as long as a is chosen properly, for example $a = 1.4\lambda$. Here we have use the fact that fact that $\ln\Lambda \sim j_{max}$ and $(aj_{max}^2)! \sim e^{2j_{max}^2\ln j_{max}}$.

So we have

$$
\begin{aligned}
|A_T|_{|T|=n} \;\; &< \;\; \int d\nu(\sigma, w) \prod_{l,l' \in \mathcal{T}} \lambda^{n-1} |\mathrm{Tr}\{||K^{\bar{\ell}}_{mn}(\sigma)||\; ||R^{\bar{\ell}_l, \bar{\ell}_r}_{mn}(\sigma)||\; |e^{\mathrm{Tr} i\sqrt{2\lambda}\sigma T_\Lambda}|\}| \\
&\times \;\; e^{\lambda \ln^3 \Lambda} e^{-a j^3_{max}} \cdot (a j^2_{max})! \leqslant (K\lambda)^{n-1}. \qquad (85)
\end{aligned}
$$

Notice that the global trace operator Tr would also result in a cutoff index Λ, but this factor could be easily bounded by the exponential $e^{-aj^3_{max}}$ when a is properly chosen, and K is an arbitrary constant times the possible factors $\sqrt{2}$ or $\sqrt{2}+1$ comes from the bound of the resolvent R and the leaf K, respectively.

This is the Nelson's argument in the formalism of Loop vertex expansion.

8 Borel summability

Theorem 8.1. *The perturbation series of the connected function for ϕ^4_2 theory is Borel summable.*

Proof For the perturbation series $\sum_{n=0} a_k \lambda^k$ to be Borel summable to the function G, we need to have

$$G(\lambda) = \sum_{n=0} a_k \lambda^k + R^{(n+1)}(\lambda), \qquad (86)$$

where $R^{(n+1)}(\lambda)$ is the Taylor remainder. The analyticity domain C_λ for G should be at least $|\lambda| < \frac{1}{K}$ and $\mathrm{Re}\lambda > 0$ [42, 27], which means

$$-\frac{\pi}{4} \leq \mathrm{Arg}\sqrt{\lambda} \leq \frac{\pi}{4}. \qquad (87)$$

We rewrite the resolvent as

$$R = \frac{1}{1 + i\sqrt{2\lambda} C^{1/2} \sigma C^{1/2}}. \qquad (88)$$

Since the matrix $C^{1/2}\sigma C^{1/2}$ is Hermitian, its eigenvalues are real, so that there are no poles in the denominator. In the analytic domain of G we have

$$\| R \| = |\frac{1}{1 + i\sqrt{2\lambda} C^{1/2} \sigma C^{1/2}}| \leqslant \sqrt{2}, \qquad (89)$$

and

$$\| K \| = \| R - 1 \| \leqslant 1 + \sqrt{2}. \qquad (90)$$

However in the analytic domain C_λ the linear counter term becomes:

$$e^{\mathrm{Tr} i\sqrt{2\lambda}\sigma T_\Lambda} = e^{\mathrm{Tr} i|\sqrt{2\lambda}|\cos\theta \sigma T_\Lambda} e^{-\mathrm{Tr}|\sqrt{2\lambda}|\sin\theta\sigma T_\Lambda}, \qquad (91)$$

where $\theta = \mathrm{Arg}\sqrt{\lambda}$. We could bound the first term in (91) by 1, but the second term would diverge for negative σ.

We rewrite this term as:

$$\int d\mu(\sigma) e^{-1/2\mathrm{Tr}\sigma^2} e^{-\mathrm{Tr}|\sqrt{2\lambda}|\sin\theta\sigma T_\Lambda}$$
$$= \int d\mu(\sigma) e^{-1/2\mathrm{Tr}(\sigma+\sqrt{2\lambda}\sin\theta T_\Lambda)^2} e^{\sin^2\theta T_\Lambda^2}. \tag{92}$$

The term $e^{\sin^2\theta T_\Lambda^2}$ could diverge at worst as $e^{1/2T_\Lambda^2}$ for $\theta = \pm\pi/4$. But this is not dangerous, since we could still bound it with the convergent factor $e^{-aj_{max}^3} \sim e^{-aT_\Lambda^3}$ that we gained from the crossings and nesting lines.

We use simply the Taylor expansion with remainder for the connected function (86):

$$G(t\lambda)|_{t=1} = \frac{G^{(n)}(\lambda)}{n!}|_{t=0} + \int_0^1 dt \frac{(1-t)^n}{n!} G^{(n+1)}(t\lambda), \tag{93}$$

followed by explicit Wick contractions. We have for the reaminder

$$||R^{n+1}|| < |\lambda|^{n+1} K^n (2n)!! \leq |\lambda|^{n+1} [K']^n (n!), \tag{94}$$

where K and K' are positive numbers including the possible factors $\sqrt{2}$ or $\sqrt{2}+1$ from the bound of the resolvent R_{mn} and of the leaf K_{mn} respectively. Hence we have proved the Borel summability of the perturbation series. □

9 Appendix

9.1 The order 1 case.

There are only three terms for order 1 expansion and the corresponding graphs are shown in Figure 7. The amplitude for graph A reads:

$$\begin{aligned} G_A^1 &= -\frac{1}{2}[-\frac{1}{2}(i\sqrt{2\lambda})^2] \int d\nu(\sigma) \sum_{mnp} [(C_{mn}\sigma_{np}C_{mp}\sigma_{pn})] \times 3 \\ &= -\frac{3}{2} T_\Lambda^2, \end{aligned} \tag{95}$$

where the factor 3 comes from the number of contractions that can form graph A. The amplitude for graph B reads:

$$G_B^1 = \frac{1}{2!}(i\sqrt{2\lambda})^2 \int d\nu(\sigma)[\mathrm{Tr}_m T_m^\Lambda \sigma_{mm} \mathrm{Tr}_n T_n \sigma_{nn}] = -\lambda T_\Lambda^2. \tag{96}$$

Graph C is the constant counter-term with amplitude $\frac{5}{2}T_\Lambda^2$. So we have

$$G_A^1 + G_B^1 + G_C^1 = 0. \tag{97}$$

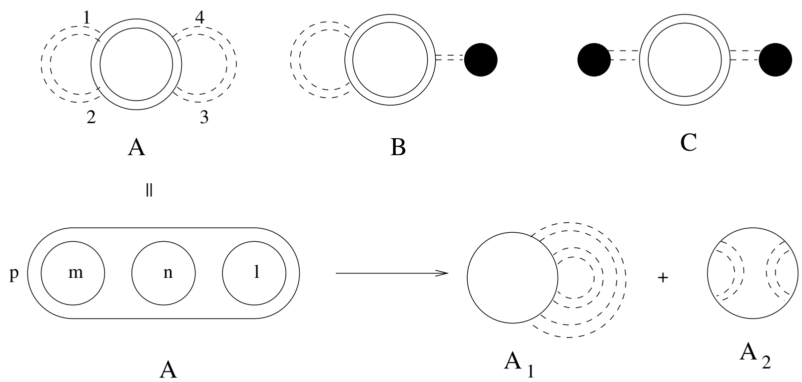

Figure 15: The renormalization of the divergent graphs at order 2.

9.2 The order 2 case

There are three kinds of graphs of order 2 which are shown in Figure 15 with nontrivial combinatorial factors. The amplitude of graphs A reads:

$$G^{(2)}_A = -\frac{1}{2}\frac{-(i\sqrt{2\lambda})^4}{4}\sum_{mnlp}^{\Lambda}\frac{1}{m+p+1}\frac{1}{n+p+1}\frac{1}{l+p+1}\times D^{(2)}_A$$
$$= T^3_\Lambda, \tag{98}$$

where

$$T^3_\Lambda = \sum_{mnlp}^{\Lambda}\frac{1}{m+p+1}\frac{1}{n+p+1}\frac{1}{l+p+1}, \tag{99}$$

and $D^{(2)}_A$ is the combinatorial factor for graph A. For generating graph A we could either contract the half line 1 with 2 and 3 with 4, or 1 with 4 and 3 with 2. If the sigma fields are attached to the inner face like graph A_2, these are all possible cases. If the σ fields are attached to the outer faces, there are two additional cases for each kind of contraction, take A_1 for example. So in total there are 8 possibilities and $D^{(2)}_A = 8$. Similarly we have $D^{(2)}_B = 9$ and $D^{(2)}_C = 4$.

So the amplitude of graphs B reads:

$$G^{(2)}_B = \frac{1}{2!}\cdot 2\cdot\left[-\frac{1}{2}\frac{(i\sqrt{2\lambda})^3}{3}(i\sqrt{2\lambda})\sum_{nl}^{\Lambda}\mathrm{Tr}_pT_p\frac{1}{n+p+1}\frac{1}{l+p+1}\right]\times D^{(2)}_B$$
$$= -\frac{3}{2}T^3_\Lambda, \tag{100}$$

and the amplitude for graphs C reads:

$$G^{(2)}_C = \frac{1}{3!}\cdot 3\cdot\left[-\frac{1}{2}\frac{-(i\sqrt{2\lambda})^2}{2}(i\sqrt{2\lambda})^2\sum_{n}^{\Lambda}\mathrm{Tr}_pT_p\mathrm{Tr}_pT_p\frac{1}{n+p+1}\right]\times D^{(2)}_C$$
$$= \frac{1}{2}T^3_\Lambda, \tag{101}$$

So again we have

$$G_A^{(2)} + G_B^{(2)} + G_C^{(2)} = 0. \tag{102}$$

This checks explicitly the renormalization at low orders.

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